If we now consider the body forces per unit mass to be negligible, the Navier-Stokes equation (14) becomes

$$0 = \frac{\partial P}{\partial r} \tag{17a}$$

$$0 = \frac{\partial P}{\partial \Theta} \tag{17b}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \tag{17c}$$

The first two of equations (17) yield

$$P = P\ (z,t) \tag{18}$$

that is, the pressure is, at most, a function of the z-coordinate and time.

The three-dimensional continuity equation in cylindrical coordinates can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho ur)}{r\partial r} + \frac{\partial(\rho v)}{r\partial \theta} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{19}$$

Combining the previous conditions of constant density and zero cross-flow with the continuity equation, we obtain

$$\frac{\partial w}{\partial z} = 0 \tag{20}$$

Thus the velocity w along the longitudinal axis of the tube is independent of the z-coordinate. Inserting this relation in equation (17c), we obtain

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{21}$$

For the case of laminar flow we consider the pressure along the longitudinal axis of the tube to vary linearly from a maximum value in the compression chamber to atmospheric pressure at the terminal point of the tube. Therefore, if we neglect end effects the pressure within the tube can be expressed in terms of time and the coordinate z as

$$P = P_g e^{-t/\Theta_1}\left(1 - \frac{z}{L}\right) + P_a \tag{22}$$

where

L . . . is length of bore (see Fig. 8), in

Placing this expression for the pressure in equation (21), we have

$$\nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{\partial w}{\partial t} = -\frac{P_g}{\rho L} e^{-t/\Theta_1} \tag{23}$$

The initial, final, and boundary conditions for the subject problem are

Initial condition: $w(r, 0) = 0, \ 0 \le r \le R_o$ (24)

Final condition: $w(r, \infty) = 0, \ 0 \le r \le R_o$ (25)

Boundary condition: $w(R_o, t) = 0, \ t \ge 0$ (26)

Equation (23) and the above boundary conditions constitute the governing boundary value problem for determining the fluid velocity within the bore of the subject knock-off tube. We now proceed to determine a particular solution of equation (23).

The homogeneous part of (23) is written as

$$\nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{\partial w}{\partial t} = 0 \tag{27}$$